For each integer \( n \geq 0 \), let \( S(n) = n - m^2 \), where \( m \) is the greatest integer with \( m^2 \leq n \). Define a sequence \( (a_k)_{k=0}^\infty \) by \( a_0 = A \) and \( a_{k+1} = a_k + S(a_k) \) for \( k \geq 0 \). For what positive integers \( A \) is this sequence eventually constant?

**Solution:** The sequence is eventually constant if and only if \( A \) is a perfect square.

We see that if at any point \( a_k \) is a perfect square, then \( S(a_k) = 0 \), and thus \( a_{k+1} = a_k + S(a_k) = a_k \).

On the other hand, if \( a_k \) is not a perfect square for some \( k \geq 0 \), then there exists a positive integer \( m \) so that \( m^2 < a_k < (m + 1)^2 \). This gives \( a_k = m^2 + t \) for some \( 1 \leq t \leq 2m \) and \( S(a_k) = t \), which yields \( a_{k+1} = m^2 + 2t \). Now we see that \( m^2 < a_k < a_{k+1} = m^2 + 2t \leq m^2 + 4m < (m+2)^2 \). Moreover, since they differ in parity, it is impossible that \( a_{k+1} = m^2 + 2t \) and \( (m+1)^2 \) are equal, thus, \( a_{k+1} \) is not a perfect square. Consequently, if \( a_0 \) is not a perfect square, then no subsequent \( a_i \) is a perfect square, and the sequence will never become constant.

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