Define the sequence \( \{a_n\} \) by \( a_1 = 1 \) and for \( n \geq 1 \), \( a_{n+1} = \cos[\arctan(a_n)] \).

Find a formula for \( a_n \) and compute \( \lim_{n \to \infty} a_n \).

**Solution:** We prove that \( a_n = \sqrt{f_{n}/f_{n+1}} \), where \( f_n \) is the \( n^{th} \) Fibonacci number, giving that

\[
\lim_{n \to \infty} a_n = \sqrt{1/\phi},
\]

where \( \phi = (1 + \sqrt{5})/2 \), the golden ratio.

First note that since \( 0 \leq \arctan(x) \leq \pi/2 \), each \( a_n \) is positive. Letting \( \Phi_n = \arctan(a_n) \), we have \( \tan(\Phi_n) = a_n \) and \( a_{n+1} = \cos(\Phi_n) \). By drawing the appropriate triangle, this gives that

\[
a_{n+1} = \frac{1}{\sqrt{1 + a_n^2}},
\]

Looking at the first few terms of the sequence in an attempt to detect a pattern, we compute

\[
a_1 = 1, \quad a_2 = \sqrt{\frac{1}{2}}, \quad a_3 = \sqrt{\frac{2}{3}},
\]

\[
a_4 = \sqrt{\frac{3}{5}}, \quad a_5 = \sqrt{\frac{5}{8}}, \quad a_6 = \sqrt{\frac{8}{13}},
\]

and from this data it appears that

\[
a_n = \sqrt{\frac{f_n}{f_{n+1}}},
\]

where \( f_n \) denotes the \( n^{th} \) Fibonacci number (\( f_1 = f_2 = 1 \) and \( f_{n+2} = f_{n+1} + f_n \)). We will use induction to show that this is indeed the case.

Let \( P(n) \) be the statement that \( a_n = \sqrt{f_n/f_{n+1}} \). Then \( P(1) \) is true since \( f_1 = f_2 = 1 \) and \( \sqrt{\frac{f_1}{f_2}} = \sqrt{\frac{1}{1}} = 1 = a_1 \) and \( P(2) \) is true since \( f_3 = 2 \) and
\[
\sqrt{\frac{f_2}{f_3}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = a_2. \text{ Now choose } m \geq 2 \text{ and assume } P(n) \text{ is true for all } n \in \{1, 2, \ldots, m\}. \text{ We then have }
\]

\[
a_{m+1} = \frac{1}{\sqrt{1 + a_m^2}} = \frac{1}{\sqrt{1 + (\sqrt{f_m/f_{m+1}})^2}} \quad \text{by the inductive hypothesis}
\]

\[
= \frac{1}{\sqrt{1 + \frac{f_m}{f_{m+1}}}}
\]

\[
= \sqrt{\frac{f_{m+1}}{f_{m+1} + f_m}}
\]

\[
= \sqrt{\frac{f_{m+1}}{f_{m+2}}}. 
\]

Therefore \(P(m+1)\) is true whenever \(P(1), P(2), \ldots, P(m)\) are all true. Thus, by induction, \(P(n)\) is true for all \(n \geq 1\). Since

\[
\lim_{n \to \infty} \frac{f_{n+1}}{f_n} = \frac{1 + \sqrt{5}}{2} \quad \text{(the golden ratio)}
\]

we have that

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \sqrt{\frac{f_n}{f_{n+1}}} = \sqrt{\frac{2}{1 + \sqrt{5}}} \approx 0.78615
\]

Solutions for this problem were submitted by Rob Hill (Gambrills, Maryland), Kipp Johnson (Beaverton, OR), Hari Kishan (India), Tom O’Neil (Central Coast of CA), Colin Perera (San Antonio), Surajit Rajagopal (India), A. Teitelman (Israel), and the Waubonssee MEC Club.