Problem of the Week #5
10/17/2016 to 10/30/2016

Find all sets of four real numbers—say $x_1, x_2, x_3, x_4$—such that the sum of any one and the product of the other three is equal to 2.

**Solution:** We are looking for solutions to

\[
\begin{align*}
x_1 + x_2 x_3 x_4 &= 2 \\
x_2 + x_1 x_3 x_4 &= 2 \\
x_3 + x_1 x_2 x_4 &= 2 \\
x_4 + x_1 x_2 x_3 &= 2,
\end{align*}
\]

which is symmetric in $x_i$. Set $p := x_1 x_2 x_3 x_4$. If $x_1 = 0$, then $x_2 x_3 x_4 = 2$, and we also have that $x_2 = x_3 = x_4 = 2$, a contradiction, so by symmetry, none of the $x_i$ are zero, and $p \neq 0$.

Rewrite the equations in our system as

\[
x_i + \frac{p}{x_i} = 2 \Rightarrow x_i^2 - 2x_i + p = 0.
\]

For each $i$, this equation has at most two solutions,

\[
x_i = 1 \pm \sqrt{1 - p},
\]

and for these solutions to be real, we need $p \leq 1$. This gives us three cases to consider.

**Case I:** The roots are equal. Then $p = 1$ and so $x_1 = x_2 = x_3 = x_4 = 1$.

**Case II:** The roots are not equal with two of the $x_i$ being $1 + \sqrt{1 - p}$ and two of them being $1 - \sqrt{1 - p}$. This yields

\[
p = (1 + \sqrt{1 - p})^2(1 - \sqrt{1 - p})^2 = p^2,
\]

which implies that $p = 1$, a contradiction.

**Case III:** The roots are not equal with one of them being distinct from the other three.

If

\[
p = (1 + \sqrt{1 - p})^3(1 - \sqrt{1 - p}) = p(1 + \sqrt{1 - p})^2,
\]

then again $p = 1$, a contradiction. If

\[
p = (1 + \sqrt{1 - p})(1 - \sqrt{1 - p})^3 = p(1 - \sqrt{1 - p})^2,
\]
then $p = 1$ or $p = -3$. Since $x_i = 1 \pm \sqrt{1 - p}$, this gives that one of the numbers is 3 and the others are -1. Thus, $(x_1, x_2, x_3, x_4)$ is one of the following sets:

$$
(1, 1, 1, 1), (3, -1, -1, -1), (-1, 3, -1, -1), (-1, -1, 3, -1), (-1, -1, -1, 3).
$$

Solutions for this problem were submitted by Mark Girard (alum), Rob Hill (Gambrills, Maryland), Lincoln James (Chicago, IL), Hari Kishan (India), Lee (Ithaca, NY), Tom O’Neil (Central Coast of CA), Benjamin Phillabaum (Northbrook, IL), Luciano Santos (Lisboa, Portugal), F. Wallner (Germany), and Yian Ann Xu (Beaverton, OR).