Problem of the Week #2
9/05/2016 to 9/18/2016

Suppose that $a, b, \ldots, n$ are distinct, positive integers, none of which is divisible by any primes greater than 3. Show that

$$\frac{1}{a} + \frac{1}{b} + \cdots + \frac{1}{n} < 3.$$ 

Solution: Let

$$S = \frac{1}{a} + \frac{1}{b} + \cdots + \frac{1}{n}.$$ 

By assumption, every term in $S$ is of the form $\frac{1}{2^r3^s}$ for some nonnegative integers $r$ and $s$. Let $t$ be the largest exponent that occurs among $r$ and $s$ over all terms in $S$ and define

$$U = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^t}$$

and

$$V = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^t}.$$ 

Then every term in $S$ is some term that can be found in the expansion of $UV$, so in particular, $S \leq UV$. Now

$$U = \frac{1 - (1/2)^{t+1}}{1 - (1/2)} = 2(1 - (1/2)^{t+1}) < 2$$

and

$$V = \frac{1 - (1/3)^{t+1}}{1 - (1/3)} = \frac{3(1 - (1/3)^{t+1})}{2} < \frac{3}{2}.$$ 

Thus, $S \leq UV < 2 \cdot (3/2) = 3$.

Solutions for this problem were submitted by M.V. Channakeshava (Bengaluru, India), Lloyd Christmas (San Antonio . . . not his real name), Mark Crawford (Sugar Grove, IL), Rob Hill (Gambrills, Maryland), Lincoln James (Chicago, IL), Kipp Johnson (Beaverton, OR), Jack Kennedy (Seattle, WA), Steve King (Pullman, WA), Hari Kishan (India), Yehuda Koslowe (Bergenfield, NJ), Tom O’Neil (Central Coast of CA), Benjamin Phillabaum (Northbrook, IL), and Luciano Santos (Lisboa, Portugal).