Problem of the Week #1
8/22/2016 to 9/04/2016

One can consider the 3-by-3 arrangements of the digits 1 through 9 that represent a sum, such as the following.

\[
\begin{array}{c}
318 \\
+ 654 \\
\hline
972
\end{array}
\]

Similarly, one can consider the 3-by-3 arrangements of the digits 1 through 9 that form a serial, connected chain when traveling one space at a time, horizontally or vertically. That is, when starting at the 1 and moving a single (up/down or left/right) space at a time, one can visit the 2, then the 3, and so on, until reaching the 9. The following arrangement is an example of such a chain.

\[
\begin{array}{ccc}
1 & 2 & 3 \\
8 & 7 & 4 \\
9 & 6 & 5
\end{array}
\]

Find all 3-by-3 arrangements that both represent a sum and form such a chain, or prove that no such arrangement exists.

Solution: We can certainly do this by brute force, but let’s make a couple of observations that will allow us to eliminate most of our options. To make for an easier discussion, imagine that we color the cells of a 3-by-3 grid like a checkerboard so that the corner squares and the central square are black.

1. We see that when we place the digits 1 through 9 on this colored grid, we alternate colors as we make a chain. Since there are five odd digits and four even digits, the only way to place the digits 1 through 9 to form a chain would be for the odd digits to be located in the black squares and the even digits to be placed in the remaining four white squares.

2. In the squares containing the even digits, the 2 and 4 cannot be opposite one another, as this would not enable the completion of a chain. Similarly, if the 2 is opposite the 8, the rest of the digits are fixed, and these do not yield a desired arrangement. So, the only possibility for a proper arrangement would be for the 2 to be opposite the 6 and the 4 to be opposite the 8.
3. The top two digits in the middle column of the grid add up to an odd number and yet the number in the bottom of the middle column is even, so the two upper digits in the right column of our grid must sum to a number greater than 10 (so that a 1 is carried).

4. The top two digits in the left column must sum up to a number less than 10. With this in mind, we can see that the following arrangement works, and in fact, it is the only such arrangement.

\[
\begin{array}{ccc}
1 & 2 & 9 \\
4 & 3 & 8 \\
5 & 6 & 7 \\
\end{array}
\]

Note: This is Problem 5 on pages 221–222 of *Martin Gardner’s New Mathematical Diversions from Scientific American*, 1966.

Solutions for this problem were submitted by Swarnabja Bhaumik (Calcutta, India), M.V. Channakeshava (Bengaluru, India), Sandipan Dey (Kolkata, India), Mark Girard (Canada, alum), Rob Hill (Gambrills, Maryland), Kipp Johnson (Beaverton, OR), Jack Kennedy (Seattle, WA), Steve King (Pullman, WA), Hari Kishan (India), Yehuda Koslowe (Bergenfield, NJ), Tom O’Neil (Central Coast of CA), Jay Pantone (Dartmouth), and Luciano Santos (Lisboa, Portugal).