If they exist, find polynomials $f(x)$, $g(x)$, and $h(x)$ such that

$$|f(x)| - |g(x)| + h(x) = \begin{cases} 
-1 & \text{if } x < -1 \\
3x + 2 & \text{if } -1 \leq x \leq 0, \text{ and} \\
-2x + 2 & \text{if } x > 0.
\end{cases}$$

Solution: We would hope that $f$, $g$, and $h$ are linear functions and that the change in description of the function $p(x) = |f(x)| - |g(x)| + h(x)$ comes from the functions $f$ and $g$ changing sign. To that end, let $f(x) = m_1x + b_1$ and assume that $f(x) \geq 0$ if $x \leq -1$ and $f \leq 0$ otherwise. Similarly, let $g(x) = m_2x + b_2$ and assume that $g(x) \geq 0$ if $x \leq 0$ and $g \leq 0$ otherwise. Also define $h(x) = m_3x + b_3$. Then, if $x < -1$ we have

$$p(x) = |f(x)| - |g(x)| + h(x) = m_1x + b_1 - m_2x - b_2 + m_3x + b_3 = -1,$$

if $-1 \leq x \leq 0$ we have

$$p(x) = |f(x)| - |g(x)| + h(x) = -m_1x - b_1 - m_2x - b_2 + m_3x + b_3 = 3x + 2,$$

and if $x > 0$ we have

$$p(x) = |f(x)| - |g(x)| + h(x) = -m_1x - b_1 + m_2x + b_2 + m_3x + b_3 = -2x + 2.$$

Equating the coefficients of $x$ and the constant terms in all three equations yields two systems of three equations and three unknowns.

$$\begin{align*}
m_1 - m_2 + m_3 &= 0 \\
-m_1 - m_2 + m_3 &= 3 \\
-m_1 + m_2 + m_3 &= -2
\end{align*}$$

and

$$\begin{align*}
b_1 - b_2 + b_3 &= -1 \\
-b_1 - b_2 + b_3 &= 2 \\
-b_1 + b_2 + b_3 &= 2.
\end{align*}$$
Solving these systems we obtain

\[ m_1 = \frac{-3}{2}, \quad m_2 = \frac{-5}{2}, \quad m_3 = -1, \quad b_1 = \frac{-3}{2}, \quad b_2 = 0, \quad \text{and} \quad b_3 = \frac{1}{2}. \]

Thus,

\[ f(x) = -\frac{3}{2}x - \frac{3}{2}, \quad g(x) = -\frac{5}{2}x, \quad \text{and} \quad h(x) = -x + \frac{1}{2} \]

is a solution.

Solutions for this problem were submitted by Harald Bensom (Oberhausen, Germany), Kipp Johnson (Beaverton, OR), Jack Kennedy (Seattle, WA), Hari Kishan (India), Tin Lam (St. Louis, MO), Tom O’Neil (Central Coast, CA), Benjamin Phillabaum (Northbrook, IL), Krishna Sambath (Houston, TX), and Luciano Santos (Portugal)).