Problem of the Week #6
11/02/2015 to 11/15/2015

Consider all lines which meet the graph of \( y = 2x^4 + 7x^3 + 3x - 5 \) in four different points, say \((x_i, y_i)\) for \(i = 1, 2, 3, 4\). Show that

\[
\frac{x_1 + x_2 + x_3 + x_4}{4}
\]

is independent of the line and find its value.

**Solution:** The desired value is \(-7/8\).

Assuming that the line \(y = mx + b\) meets the quartic’s graph at the four points given, we know that \(x_1, x_2, x_3, x_4\) will be the roots of the quartic equation

\[
2x^4 + 7x^3 + 3x - 5 - (mx + b) = 2x^4 + 7x^3 + (3 - m)x - 5 - b = 0,
\]

that is,

\[
2 \left( x^4 + \frac{7}{2}x^3 + \frac{3-m}{2}x - \frac{5+b}{2} \right) = 2(x - x_1)(x - x_2)(x - x_3)(x - x_4).
\]

Expanding the right side of this equation and then equating the powers of \(x^3\), we see that \(x_1 + x_2 + x_3 + x_4 = -\frac{7}{2}\) so that

\[
\frac{x_1 + x_2 + x_3 + x_4}{4} = -\frac{7}{8},
\]

regardless of the values of \(m\) and \(b\).

Solutions for this problem were submitted by Harald Bensom (Oberhausen, Germany), M.V.Channakeshava (Bengaluru, India), Mark Crawford (Sugar Grove, IL), Sandipan Dey (Kolkata, India), Kipp Johnson (Beaverton, OR), Jack Kennedy (Seattle, WA), Steve King (Pullman, WA), Hari Kishan (India), Yehuda Koslowe (Bergenfield, NJ), Tin Lam (St. Louis, MO), Tom O’Neil (Central Coast, CA), Doug Ray (TX State), Krishna Sambath (Houston, TX), and Dennis Ugolini (TU).