Problem of the Week #4
10/05/2015 to 10/18/2015

Find all triples \((a, b, c)\) such that \(ab - c\), \(bc - a\), and \(ca - b\) are all powers of two.

Solution: It turns out that this was a difficult problem, so kudos to those of you who attempted it (and extra kudos if you succeeded in an actual solution). While it was not specified, let’s just assume that \(a, b, c\) are positive integers to make the solution somewhat tenable. (For instance, Brandon Jeong pointed out that \((-1, -1, -1)\) would be such a triple if we let our triples include any integers.) In the positive integer case, the triples we desire are any permutation of the following four:

\[(2, 2, 2), (2, 2, 3), (2, 6, 11), (3, 5, 7).\]

To begin, suppose that \(a \leq b \leq c\). Now there exist \(i, j, k \geq 0\) so that

\[ab - c = 2^i, \quad (1)\]
\[ac - b = 2^j, \quad (2)\]
\[bc - a = 2^k. \quad (3)\]

Given the relationship between \(a, b,\) and \(c\), we can use Equations (1)-(3) to show that \(i \leq j \leq k\).

If \(a = 1\), then \(b - c = 2^i\) and \(c - b = 2^j\), but since \(2^i, 2^j > 0\), this is impossible. Accordingly, \(a > 1\). Now suppose that \(a = b \geq 3\). Then \(2^j = ac - a = a(c - 1)\). Since \(c \geq a, b \geq 3, a, c - 1 > 1\) are power of 2 which are bigger than 1, giving that \(a\) is even and \(c\) is odd. Thus, \(ab - c = a^2 - c = 2^i = 1\). Then \(a^2 = c + 1\) is also a power of 2, so \(c = 3\); however, \(a = b = c = 3\) is not a solution to the problem, as \(3^3 - 3 = 6\) is not power of 2.

Now assume that \(a = 2\). Then \(2b - c = 2^i, 2c - b = 2^j,\) and \(bc - 2 = 2^k\). Since \(c \geq a = 2, 2c - b = 2^j \geq 2\), giving us that \(b\) is even (say \(b = 2s\)). If \(k = 1\), then \(c = 2,\) and checking this, we see that \((2, 2, 2)\) is a triple that works. Now suppose \(k > 1\). Then \(bc = 2sc = 2^k - 2\), giving that \(sc = 2^{k-1} - 1\) which is odd, forcing \(c\) to be odd. Thus, \(2b - c = 2^i = 1\), and playing with Equations (1)-(3) yields \(3b = 2^j + 2\) (so \(j \geq 2\)), \(3c = 2^j + 1\), and \((2^j + 1)(2^{j+1} + 1) = 9(2^{k-1} + 1)\). Using this last equation gives us that \(1 \equiv 9 \mod 2^{j-1} \Rightarrow j \in \{2, 4\}\). When \(j = 2\) we get a corresponding triple of \((2, 2, 3)\), and when \(j = 4\) we get the triple \((2, 6, 11)\), both of which work.

Finally, assume that \(3 \leq a < b \leq c\). Since \(a(c - 1) = ac - c \leq ac - b = 2^j, c \leq 2^{j-1}\). Now, \(b + a < 2c \leq 2^{j+1}/(a - 1) \leq 2^j\) and \(b - a < c \leq 2^{j-1}\), so that \(b - a\) is not divisible by \(2^{j-1}\), and if \(a > 4\), then \(b + a\) is not divisible by \(2^{j-1}\). Adding and subtracting Equations (2)
and (3) gives us that 
\[(c - 1)(b + a) = 2^k + 2^j\] and 
\[(c + 1)(b - a) = 2^k - 2^j\]. Since \(b + a < 2^{j-1}\), 
c + 1 is divisible by 4, and consequently, \(c - 1\) is not divisible by 4. Then \(b + a < 2^j\) is 
a multiple of \(2^{j-1}\), i.e.,
\[b + a = 2^{j-1}.\] (4)

When \(a = 3\), Equations (1), (2), and (4) give us that
\[3b - c = 2^i,\] 
\[3c - b = 2^j,\] and 
\[b = 2^{i-1} - 3.\]

Combining the last two of these equations, we get that \(c = b + 2\), and substituting this back 
into the first equation give us the triple \((3, 5, 7)\), which works.

Finally when \(a = 4\), Equations (2) and (4) give us that
\[4c - b = 2^i,\] and 
\[b = 2^{i-1} - 4.\]

Solving gives that
\[c = 3 \cdot 2^{i-3} - 1 \geq b = 2^{i-1} - 4 \Rightarrow 2^{i-3} \leq 3,\]
and the fact that \(a = 4 < b = 2^{i-1} - 4\) gives that \(2^{i-3} > 2\). Since no such \(j\) exists, there are 
no triples with \(a = 4\).

Solutions for this problem were submitted by Harald Bensom (Oberhausen, Germany), R. Govindan (Chennai, India), Brandon Jeong (Beaverton, OR), Steve King (Pullman, WA), Hari Kishan (India), Tom O’Neil (Central Coast, CA), and Michael Tomaine (Bellevue, WA).