Problem of the Week #2
09/07/2015 to 09/20/2015

Given a positive, three-digit integer \( n \), define \( f(n) \) to be the sum of the three digits of \( n \), their three products in pairs, and the product of all three digits (e.g., \( f(625) = 6 + 2 + 5 + 6 \cdot 2 + 6 \cdot 5 + 2 \cdot 5 + 6 \cdot 2 \cdot 5 = 125 \)). Find all positive, three-digit integers \( n \) such that \( n = f(n) \).

Solution: The only numbers are 199, 299, 399, 499, 599, 699, 799, 899, and 999.

Think of \( n = abc \), to mean that \( 1 \leq a \leq 9 \) and \( 0 \leq b, c \leq 9 \) and \( n = 100a + 10b + c \). Then \( f(n) = a + b + c + ab + ac + bc + abc \). Thus,

\[
f(n) = n \ \Rightarrow \ 99a + 9b = abc + ab + bc + ac
\]
\[
\Rightarrow \ (9 - c)b = a(bc + b + c - 99).
\]

Since \( b, c \leq 9 \), \( bc + b + c - 99 \leq 0 \), and since \( a > 0 \), \( a(bc + b + c - 99) \leq 0 \). However, \( (9 - c)b \geq 0 \) Thus, equality can only occur when both sides are 0. Since \( a \neq 0 \), it must be that \( bc + b + c - 99 = 0 \), which only occurs when \( b = c = 9 \). Thus, the numbers for which \( \frac{n}{f(n)} = 1 \) are 199, 299, 399, 499, 599, 699, 799, 899, and 999.

Solutions for this problem were submitted by Swarnabja Bhaumik (West Bengal, India), Sandipan Dey (India), James Guerry (Bell, FL), Álvaro Jiménez (Madrid, Spain), Kipp Johnson (Beaverton, OR), Jack Kennedy (Seattle, WA), Dayton King (alum), Hari Kishan (India), Yehuda Koslowe (Bergenfield, NJ), Tin Lam (St. Louis, MO), Jason Lee (Maryland), Tom O’Neil (Central Coast, CA), Sean Pan (TU), Benjamin Phillabaum (Northbrook, IL), Raquel Sánchez (Madrid, Spain), and Dennis Ugolini (TU).